CHAPTER 3:

Bayesian Decision Theory

Probability and Inference

- Result of tossing a coin is $\in$ {Heads,Tails}
- Random var $X \in \{1,0\}$

  Bernoulli: $P\{X=1\} = p_o^{X} (1 - p_o)^{(1-X)}$
- Sample: $\mathbf{X} = \{x^t\}_{t=1}^{N}$

  Estimation: $p_o = \#\{\text{Heads}\}/\#\{\text{Tosses}\} = \sum_t x^t / N$
- Prediction of next toss:

  Heads if $p_o > ½$, Tails otherwise

Classification

- Credit scoring: Inputs are **income** and **savings**.
  Output is **low-risk** vs **high-risk**
- Input: $\boldsymbol{x} = [x_1, x_2]^T$,Output: C $\in$ {0,1}
- **Prediction**:

$$\text{choose} \begin{cases} C = 1 \text{ if } P(C = 1 | x_1, x_2) > 0.5 \\ C = 0 \text{ otherwise} \end{cases}$$

or

$$\text{choose} \begin{cases} C = 1 \text{ if } P(C = 1 | x_1, x_2) > P(C = 0 | x_1, x_2) \\ C = 0 \text{ otherwise} \end{cases}$$

Bayes' Rule

prior *likelihood*

posterior

$$P(C|\mathbf{x}) = \frac{P(C)p(\mathbf{x}|C)}{p(\mathbf{x})}$$

evidence

$$P(C=0)+P(C=1)=1$$
$$p(\mathbf{x})=p(\mathbf{x}|C=1)P(C=1)+p(\mathbf{x}|C=0)P(C=0)$$
$$p(C=0|\mathbf{x})+P(C=1|\mathbf{x})=1$$

Bayes' Rule: *K*>2 Classes

$$P(C_i \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid C_i) P(C_i)}{p(\mathbf{x})}$$

$$= \frac{p(\mathbf{x} \mid C_i) P(C_i)}{\sum_{k=1}^{K} p(\mathbf{x} \mid C_k) P(C_k)}$$

$P(C_i) \geq 0$ and $\sum_{i=1}^{K} P(C_i) = 1$

choose C_i if $P(C_i \mid \mathbf{x}) = \max_k P(C_k \mid \mathbf{x})$

Naïve Bayes Classifier

- $p(\mathbf{x}|C)$?
  - chain rule:
    - $= p(x_1|C)\, p(x_2|x_1,C)\, p(x_3|x_1,x_2,C) \ldots p(x_n|x_1,x_2,\ldots,x_{n-1},C)$
  - Bayesian network
    - $x_3 \leftarrow x_2, \ldots, x_n \leftarrow x_k$
    - $= p(x_1|C)\, p(x_2|x_1,C)\, p(x_3|x_2,C) \ldots p(x_n|x_k,C)$
  - "Naïve" conditional independence assumption
    - $= p(x_1|C)\, p(x_2|C)\, p(x_3|C) \ldots p(x_n|C)$

Losses and Risks

- Actions: α_i
- Loss of α_i when the state is C_k : λ_{ik}
- Expected risk (Duda and Hart, 1973)

$$R(\alpha_i \mid \mathbf{x}) = \sum_{k=1}^{K} \lambda_{ik} P(C_k \mid \mathbf{x})$$

$$\text{choose } \alpha_i \text{ if } R(\alpha_i \mid \mathbf{x}) = \min_k R(\alpha_k \mid \mathbf{x})$$

Losses and Risks: 0/1 Loss

$$\lambda_{ik} = \begin{cases} 0 \text{ if } i = k \\ 1 \text{ if } i \neq k \end{cases}$$

$$R(\alpha_i \mid \mathbf{x}) = \sum_{k=1}^{K} \lambda_{ik} P(C_k \mid \mathbf{x})$$

$$= \sum_{k \neq i} P(C_k \mid \mathbf{x})$$

$$= 1 - P(C_i \mid \mathbf{x})$$

*For **minimum risk**, choose the **most probable class***

Losses and Risks: Reject

$$\lambda_{ik} = \begin{cases} 0 & \text{if } i = k \\ \lambda & \text{if } i = K+1, \quad 0 < \lambda < 1 \\ 1 & \text{otherwise} \end{cases}$$

$$R(\alpha_{K+1} \mid \mathbf{x}) = \sum_{k=1}^{K} \lambda P(C_k \mid \mathbf{x}) = \lambda$$

$$R(\alpha_i \mid \mathbf{x}) = \sum_{k \neq i} P(C_k \mid \mathbf{x}) = 1 - P(C_i \mid \mathbf{x})$$

choose C_i if $P(C_i \mid \mathbf{x}) > P(C_k \mid \mathbf{x}) \; \forall k \neq i$ and $P(C_i \mid \mathbf{x}) > 1 - \lambda$

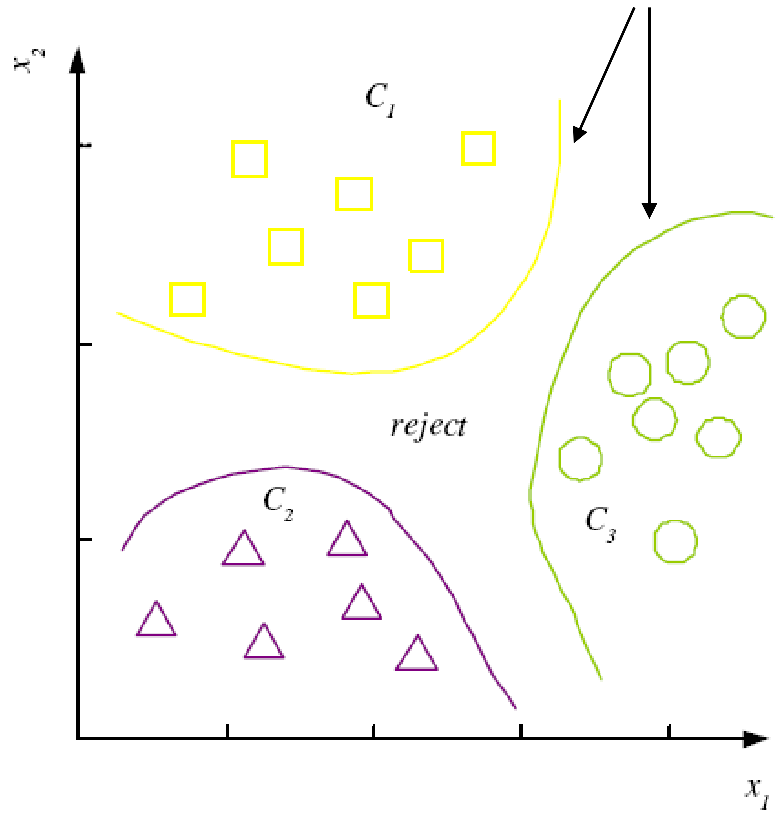
reject otherwise

Discriminant Functions

$$\text{choose } C_i \text{ if } g_i(\mathbf{x}) = \max_k g_k(\mathbf{x})$$

$$g_i(\mathbf{x}), i = 1, \ldots, K$$

$$g_i(\mathbf{x}) = \begin{cases} -R(\alpha_i \mid \mathbf{x}) \\ P(C_i \mid \mathbf{x}) \\ p(\mathbf{x} \mid C_i)P(C_i) \end{cases}$$



K decision regions $\mathcal{R}_1, \ldots, \mathcal{R}_K$

$$\mathcal{R}_i = \{\mathbf{x} \mid g_i(\mathbf{x}) = \max_k g_k(\mathbf{x})\}$$

K=2 Classes

- $g(\boldsymbol{x}) = g_1(\boldsymbol{x}) - g_2(\boldsymbol{x})$

$$\text{choose} \begin{cases} C_1 \text{ if } g(\mathbf{x}) > 0 \\ C_2 \text{ otherwise} \end{cases}$$

- *Log odds:*

$$\log \frac{P(C_1 \mid \mathbf{x})}{P(C_2 \mid \mathbf{x})}$$

Association Rules

- Association rule: $X \rightarrow Y$
- *People who buy/click/visit/enjoy X are also likely to buy/click/visit/enjoy Y.*
- A rule implies association, not necessarily causation.

Association measures

- **Support** $(X \rightarrow Y)$:

$$P(X,Y) = \frac{\#\{\text{customers who bought } X \text{ and } Y\}}{\#\{\text{customers}\}}$$

- **Confidence** $(X \rightarrow Y)$:

$$P(Y \mid X) = \frac{P(X,Y)}{P(X)}$$

$$= \frac{\#\{\text{customers who bought } X \text{ and } Y\}}{\#\{\text{customers who bought } X\}}$$

- Lift $(X \rightarrow Y)$:

$$= \frac{P(X,Y)}{P(X)P(Y)} = \frac{P(Y \mid X)}{P(Y)}$$

Apriori algorithm (Agrawal et al., 1996)

- For (X,Y,Z), a 3-item set, to be frequent (have enough support), (X,Y), (X,Z), and (Y,Z) should be frequent.
- If (X,Y) is not frequent, none of its supersets can be frequent.
- Once we find the frequent *k*-item sets, we convert them to rules: X, Y $\rightarrow$ Z, ...

  and X $\rightarrow$ Y, Z, ...